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A Comparison Study Between a Chebyshev Collocation Method and the Adomian Decomposition Method for Solving Linear System of Fredholm Integral Equations of the Second Kind

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Abstract. In this paper, we obtain the approximate solutions of a system of linear integral equations by using both Chebyshev polynomials method and Adomian decomposition method. Also, a comparison between the two methods is discussed.

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1.Introduction

Systems of integral equations have attracted much attention in a variety of applied sciences. These systems were formally derived to describe many problems in plane elastic deformation, fluid mechanics, and mixed boundary value problems in physics and engineering[1].

Approximate solutions of system of linear integral equations are of importance in physical problems. So far there exists no general method for finding solution of this problem, there is not much study on solution methods of integral equation systems, for this reason, we present in this paper two methods for solving this system, we use the Chebyshev polynomials method[2,3]and Adomian decomposition method[4,5].

2. Chebyshev Polynomials Solutions.

Different kinds of polynomials and quadrature rules play an important role in applied mathematics, a Chebyshev collocation method has been presented to solve systems of linear integral equations in terms of Chebyshev polynomials in [2]. The method transforms the integral system into the matrix equation with the help of Chebyshev points[6].

This procedures are stable and convergent, which have been proved in[7], and error estimates in weighted L^p norm, $1 \le p \le +\infty$ are given .

2.1 Chebyshev Polynomials Solutions.

We consider systems of K linear integral equations of Fredholm in the form

$$\sum_{j=1}^{k} p_{ij}(x) y_j(x) = f_i(x) + \int_{-1}^{1} \sum_{j=1}^{k} k_{ij}(x,t) y_i(t)\, dt \ , \ i = 1,2,...,k \qquad (1)$$

A system of equations can always be written as a single vector valued equations

$$\mathrm{Y}(t) = (y_1(t), y_2(t), ..., y_k(t))^T$$

Similarly , let $\boldsymbol{f}(\boldsymbol{x})$ be a vector valued function with components $f_i(x)$ and let $\boldsymbol{P}(\boldsymbol{x})$ and $\boldsymbol{K}(\boldsymbol{x},\boldsymbol{t})$ be $k \times k$ matrices with entries $p_{ij}(x)$ and $k_{ij}(x,t)$ respectively .

Then (1) reduce to :

$$\boldsymbol{P}(\boldsymbol{x})\ \boldsymbol{Y}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \int_{-1}^{1} \sum_{\boldsymbol{j}=1}^{\boldsymbol{k}} \boldsymbol{K}(\boldsymbol{x},\boldsymbol{t})\ \boldsymbol{Y}(\boldsymbol{t})\ \boldsymbol{dt} \qquad (2)$$

The purpose of this section is to get solution as truncated Chebyshev series defined by

$$y_i(t) = \sum_{j=0}^{N} a_{ij}\ \ T_j(t)\ \ , \qquad i = 1,2,3,...,k\ \ ,\ -1 \le t \le 1 \qquad (3)$$

Where $T_j(t)$ denote the Chebyshev polynomials of the first kind , a_{ij} are unknown Chebyshev coefficients and N is chosen any positive integer .

We suppose that the kernel functions and solutions of these systems can be expressed as a truncated Chebyshev series , then (3) can be written in the matrix form

$$y_i(t) = T(t)\quad A_i \qquad , \quad i = 1,2,3,...,k \qquad (4)$$

Where

$$T(t) = [T_0(t) \quad T_1(t)\ \ ... \quad T_N(t)]\ \ ,\ A_i = [a_{i0} \quad a_{i1}\ \ ...\ \ a_{iN}]^T$$

Hence, the matrix $Y(t)$ defined as a column matrix of unknown functions can be expressed by

$$Y(t) = T(t) \quad A \tag{5}$$

So that

$$T(t) = \begin{bmatrix} T(t) & 0 & . & . & . & 0 \\ 0 & T(t) & & & & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & . & . & . & T(t) \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ A_2 \\ . \\ . \\ . \\ A_k \end{bmatrix}_{k\times 1}$$

Similarly , kernel functions $k_{ij}(x,t)$ can be expanded to univariate chebyshev series for each x_s in the form

$$k_{ij}(x_s,t) = \sum_{r=0}^{N}{}'' k_r^{ij}(x_s)\ T_r(t)$$

Where a summation symbol with double primes denotes a sum with first and last terms halved , x_s are Chebyshev collocation points defined by

$$x_s = \cos\left(\frac{s\pi}{N}\right), \quad s = 0,1,2,...,N$$

And the Chebyshev coefficients $k_r^{ij}(x_s)$ are determined by means of the relation

$$k_r^{ij}(x_s) = \frac{2}{N} \sum_{m=0}^{N}{}'' k_{ij}(x_s,t_m)\ T_r(t_m)\ , \ t_m = \cos\left(\frac{m\pi}{N}\right)$$

Then the matrix representation of $k_{ij}(x_s,t)$ becomes

$$k_{ij}(x_s,t) = k_{ij}(x_s)T(t)^T \tag{6}$$

Where

$$k_{ij}(x_s) = [\tfrac{1}{2}k_0^{ij}(x_s) \quad k_1^{ij}(x_s)\ k_2^{ij}(x_s)... \quad k_{N-1}^{ij}(x_s) \quad \tfrac{1}{2}k_N^{ij}(x_s)]_{1\times(N+1)}$$

Now, in substituting the Chebyshev collocation points into Eq.(2) it obtained a matrix system. This system can be rearranged in a new matrix form

$$P\ \ Y = F + I \tag{7}$$

In which $I(x)$ denotes the integral part of Eq.(2) and

$$P = \begin{bmatrix} p(x_0) & 0 & . & . & . & 0 \\ 0 & p(x_1) & & & & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & . & . & . & p(x_N) \end{bmatrix}, \quad Y = \begin{bmatrix} y(x_0) \\ y(x_1) \\ . \\ . \\ . \\ y(x_N) \end{bmatrix}$$

$$F=\begin{bmatrix} f(x_0)\\ f(x_1)\\ .\\ .\\ .\\ f(x_N)\end{bmatrix} \quad , \quad I=\begin{bmatrix} I(x_0)\\ I(x_1)\\ .\\ .\\ .\\ I(x_N)\end{bmatrix}$$

When Chebyshev collocation points is put in relation(5) , the matrix Y becomes

$$Y=T\ \ A \tag{8}$$

Where

$$T(t)=[T(x_0)\quad T(x_1)\quad ...\quad T(x_N)]^T$$

In similar way, substituting the relation(8) in

$$I_j(x_s)=\int_{-1}^{1}\sum_{j=1}^{k}k_{ij}(x_s,t)y_i(t)\,dt\ ,\ i=1,2,...,k \tag{9}$$

And using the relation

$$z=\int_{-1}^{1}T(t)^T T(t)\ \ dt=\left[\int_{-1}^{1}T_i(t)\ T_j(t)\ \ dt\right]=\left[Z_{ij}\right]\ i,j=0,1,2,3,...,N$$

Whose entries are given in [4] as

$$Z_{ij}=\begin{cases}\dfrac{1}{1-(i+j)^2}+\dfrac{1}{1-(i-j)^2} & ,\quad for\ \ even\ \ i+j\\ \\ 0 & ,\quad for\ \ odd\ \ \ i+j\end{cases}$$

We have

$$I_i(x_s)=\sum_{j=1}^{k}\ k_{ij}(x_s)\ Z\ \ A_j\ \ ,\ i=1,2,3,...,k$$

Or

$$I(x_s)=k(x_s)\ Z\ \ A \tag{10}$$

Where

$$I(x_s)=\begin{bmatrix} I_1(x_s) \\ I_2(x_s) \\ . \\ . \\ . \\ I_k(x_s) \end{bmatrix}, \quad k(x_s)=\begin{bmatrix} k_{11}(x_s) & k_{12}(x_s) & . & . & . & k_{1k}(x_s) \\ k_{21}(x_s) & k_{22}(x_s) & & & & k_{2k}(x_s) \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ k_{k1}(x_s) & k_{k2}(x_s) & . & . & . & k_{kk}(x_s) \end{bmatrix}$$

$$Z=\begin{bmatrix} Z & 0 & . & . & . & 0 \\ 0 & Z & & & & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & . & . & . & Z \end{bmatrix}$$

Therefore, we get the matrix I in terms of Chebyshev coefficients matrix in the form

$$I = k \quad Z \;\; A \tag{11}$$

Where $\boldsymbol{k}$ is a column matrix of elements $k(x_i)$, $i = 1,2,..., N$.

Finally using the relation(8)and(11)and then simplifying(7)we have the fundamental matrix equation

$$(PT - KZ)A = F \tag{12}$$

Which corresponds to a system of $k(N+1)$ linear algebraic equations with the unknown Chebyshev coefficients. Thus unknown coefficients a_{ij} can be easily computed from this fundamental equation and thereby we find the solution of Fredholm integral system in the truncated Chebyshev series .

2.2 Numerical Examples

The method presented in this section is illustrated by the following examples.

Example 1 : Consider a system of linear Fredholm integral equations

$$y_1(x) - 2xy_2(x) = 22x + 3 + 3\int_{-1}^{1}(x+t)\; y_1(t)\; dt \;+ 3\int_{-1}^{1}(x-t)\, y_2(t)\; dt$$

$$5y_1(x) + y_2(x) = -x + 9 + 3\int_{-1}^{1}x^2\; y_1(t)\; dt \;+ 3\int_{-1}^{1}(xt + t^2)\, y_2(t)\; dt$$

And approximate the solution by truncated Chebyshev series in the form

$$y_i(x) = \sum_{j=0}^{3} a_{ij} \quad T_j(x) \quad , \; i = 1,2 \; , \qquad -1 \le x \le 1$$

Then Chebyshev collocation points are

$$x_0 = 1 \quad , \quad x_1 = \tfrac{1}{2} \quad , \quad x_2 = -\tfrac{1}{2} \quad , \quad x_3 = -1$$

After some ordinary operations it can be found Chebyshev coefficients as

$$a_{10} = 2 \quad , \quad a_{11} = 0 \quad , \quad a_{12} = 1 \quad , \quad a_{13} = 0$$

$$a_{20} = -4 \quad , \quad a_{21} = 1 \quad , \quad a_{22} = 0 \quad , \quad a_{23} = 0$$

Thus , the solution of this system is

$$y_1 = 2x^2 + 1 \qquad , \qquad y_2 = x - 4$$

Example 2 :

Consider a system of linear Fredholm integral equations

$$y_1(x) = \frac{x}{18} + \frac{17}{36} + \frac{1}{3}\int_0^1 (x+t)\; y_1(t)\; dt + \frac{1}{3}\int_0^1 (x+t)\, y_2(t)\; dt$$

$$y_2(x) = x^2 - \frac{19}{2}x + 1 + \int_0^1 x\; t\;\; y_1(t)\; dt + \int_0^1 (xt)\, y_2(t)\; dt$$

Because the integrals are bounded in the range [0,1] , then solution can be obtained by means of the shifted Chebyshev polynomials $T_j^*(t)$. So we approximate the solution by truncated Chebyshev series in the form

$$y_i(x) = \sum_{j=0}^{2} a_{ij} \quad T_j^*(x) \quad , \; i = 1,2 \; , \qquad 0 \le x \le 1$$

Then shifted Chebyshev collocation points are

$$x_0 = 1 \quad , \quad x_1 = \tfrac{1}{2} \quad , \quad x_2 = 0$$

After simple algebraic manipulation the solution of the system is given by

$$y_1 = x + 1 \qquad , \qquad y_2 = x^2 + 1$$

3. Adomian Decomposition Solution.

Recently a great deal of interest has been focused on the applications of the Adomian decomposition method to solve a wide variety of stochastic and deterministic problems[8], The solution is the sum of an infinite series which converges rapidly to the accurate solution. Recently, the Adomian decomposition method has been applied for solving systems of linear and nonlinear Fredholm integral equation of the second kind [4]. In this paper we also used the modified decomposition method .

3.1 Method of Solution.

The system of Fredholm integral equations can be written as the following

$$F(t) = G(t) + \int_a^b V(s,t,F(s))\, ds \quad , \quad t \in [a,b] \tag{13}$$

We suppose that the system(1) has a unique solution. However, the necessary and sufficient conditions for existence and uniqueness of the solution of the system(13) could be found in [1] .

Consider the ith equation of (13)

$$f_i(t) = g_i(t) + \int_a^b v_i(s,t,f_1(s),f_2(s),\ldots,f_n(s))\, ds \tag{14}$$

The canonical form of the Adomian equations can be written as

$$f_i(t) = g_i(t) + N_i(t) \tag{15}$$

Where

$$N_i(t) = N_i(f_1,f_2,\ldots,f_n)(t) = \int_a^b v_i(s,t,f_1(s),f_2(s),\ldots,f_n(s))\, ds \tag{16}$$

To use the Adomian decomposition method , Let

$$f_i(t) = \sum_{m=0}^{\infty} f_{im}(t) \qquad and \qquad N_i(t) = \sum_{m=0}^{\infty} A_{im}$$

Where $A_{im} \quad , m = 0,1,2,\ldots$ are called Adomian polynomials .

Hence (15) can be rewritten as

$$\sum_{m=0}^{\infty} f_{im}(t) = g_i(t) + \sum_{m=0}^{\infty} A_{im}(f_{10}, f_{11},\ldots, f_{1m},\ldots, f_{n0},\ldots, f_{nm}) \tag{17}$$

From (17) we define

$$f_{i0}(t) = g_i(t)$$

$$f_{i,m+1}(t) = A_{im}(f_{10},\ldots, f_{1m},\ldots, f_{n0},\ldots, f_{nm}) \; , i = 1,2,\ldots,n \quad , \quad m = 0,1,2,\ldots$$

In practice, all terms of the series $f_i(t) = \sum_{m=0}^{\infty} f_{im}(t)$ cannot be determined and so we have an approximation of the solution by the following truncated series $\varphi_{ik}(t) = \sum_{m=0}^{k-1} f_{im}(t)$ with

$$\lim_{k\to\infty} \varphi_{ik}(t) = f_i \tag{18}$$

To determine the Adomian polynomials , we write

$$f_{i\lambda}(t) = \sum_{m=0}^{\infty} f_{im}(t)\ \lambda^m \tag{19}$$

$$N_{i\lambda}(f_1, f_2, \ldots, f_n) = \sum_{m=0}^{\infty} A_{im} \quad \lambda^m \tag{20}$$

Where λ is a parameter introduced for convenience .
From (20) we obtain

$$A_{im}(t) = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} N_{i\lambda}(f_1, f_2, \ldots, f_n)\right]_{\lambda=0} \tag{21}$$

We considered $v_i(s,t,f_1(s),f_2(s),\ldots,f_n(s))$ is a linear function , Eq. (4) would be in the following form

$$N_i(t) = \int_a^b \sum_{j=1}^{n} v_{ij}(s,t)\ f_j(t)\ ds \tag{22}$$

And from (19) , (21) and (22) we get

$$A_{im}(f_{10},\ldots,f_{1m},\ldots,f_{n0},\ldots,f_{nm}) = \int_a^b \sum_{j=1}^{n} v_{ij}(s,t)\ \frac{1}{m!}\left[\frac{d^m}{d\lambda^m}\sum_{l=0}^{\infty} f_{jl}\lambda^i\right]_{\lambda=0} ds$$

$$= \int_a^b \sum_{j=1}^{n} v_j(s,t)\ f_{j,m}\ \ ds \tag{23}$$

So (17) will be as follows

$$f_{i0}(t) = g_i(t)$$

$$f_{i,m+1}(t) = \int_a^b \sum_{j=1}^{n} v_{ij}(s,t)\ f_{j,n}(t)\ \ ds\ \ ,\ i = 1,2,\ldots,n \qquad ,\quad m = 0,1,2,\ldots\ldots\ldots \tag{24}$$

The modified form was established based on the assumption that the function $g(t)$ can be divided into two parts , namely $g_0(t)$ and $g_1(t)$. Under this assumption we set

$$g(t) = g_0(t) + g_0(t)$$

Consequently , the modified recursive relation

$$f_0(t) = g_0(t)$$

$$f_1(t) = g_1(t) - L^{-1}(N\ \ f_0\ ) - L^{-1}(A_0)$$

$$f_{k+2}(t) = -L^{-1}(N\ \ f_{k+1}\ ) - L^{-1}(A_{k+1}) \qquad , \qquad k \geq 0$$

3.2 Numerical Examples

Example(3):

Consider a system of linear Fredholm integral equations in example.(2)

Adomian decomposition method for this problem consist of the following scheme

$$f_0(t) = \frac{t}{18} + \frac{17}{36} \quad , \quad g_0(t) = t^2 - \frac{19}{2}t + 1$$

$$f_{n+1}(t) = \int_0^1 (\frac{s+t}{3}) \ A_n(f_0,...,f_n,g_0,...,g_n)ds$$

$$g_{n+1}(t) = \int_0^1 s\,t \ B_n(f_0,...,f_n,g_0,...,g_n)ds$$

The modified decomposition method yields

$$f_0(t) = \frac{t}{18} \quad , \quad g_0(t) = t^2$$

$$f_1(t) = \frac{17}{36} + \int_0^1 (\frac{s+t}{3}) \left(\frac{s}{18} + s^2\right)ds = 0.1203704t + 0.561726$$

$$g_1(t) = \frac{-19}{12} + 1 + \int_0^1 s\ t \left(\frac{s}{18} + s^2\right)ds = -1.31481148t + 1$$

Table: Approximated solution for some values of t are presented in Table1.

t	$f(t)$	$\varphi_{1,11}(t)$	E_1	$g(t)$	$\varphi_{1,11}(t)$	E_2
0	1	0.988498	0.0115	1	1	0
0.2	1.2	1.184766	0.0152	1.04	1.033099	0.0069
0.4	1.4	1.381033	0.0189	1.16	1.146198	0.0138
0.6	1.6	1.577301	0.0226	1.36	1.339296	0.0207
0.8	1.8	1.773569	0.0264	1.64	1.612695	0.0276
1	2	1.969836	0.302	2	1.965494	0.0345

4. Conclusions

In this work, we conducted a comparative study between the Chebyshev collocation method and the modified decomposition method. the Chebyshev method is useful for acquiring the solution as demonstrated in examples. An interesting feature of this method is that when an integral system has linearly independent polynomial solution of degree ***N*** or less than ***N***, the method can be used for finding the Semi-analytical solution. Moreover, it is seen that when the truncation limit ***N*** is increased , there exists solution .

This method is based on computing the coefficients in Chebyshev expansion of solution of a linear integral system, and is valid when the matrix functions ***P(x)*** and ***f(x)*** are defined in [-1,1] and the kernel functions in ***k(x,t)*** have a Chebyshev series expansion in this range .

A considerable advantage of the method is that the Chebyshev coefficients of the solution are found very easily by using the computer programs such as Mathematica. Furthermore , the values of the solution at the collocation points are evaluated with the aid of the computer programs without any computational effort.

The use of the Adomian decomposition method , both for systems of linear and nonlinear Fredholm integral equations of the second kind, drive a good approximation to the solution but with a large number of iterations, when we use the modified method , as it can be seen in examples , gives better approximations , in less iteration .

The modified decomposition method is implemented in straightforward manner and it accelerates the rapid convergence of the decomposition series solution.

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References

[1] **Delves,L.M.** and **Mohamed, J.L.**,*Computational Methods for Integral Equations*, Cambridge University Press, (1985).

[2] **Aysequl Akyuz-Dascioglu** , Chebyshev Polynomial Solutions of Systems of Linear Integral Equations, *Applied Mathematics and Computation*, **151**:221-232 (2004).

[3] **Elcin Yusufogiu** (*AGADJANOV*), Numerical Expansion Methods for Solving Systems of Linear Integral Equations Using Interpolation and Quadrature Rules, *International Journal of Computer Mathematics*, **84**(1):133-149:(2007).

[4] **Babolian,E,Biazar,J.** and **Vahidi,A.R.,** The Decomposition method applied to systems of Fredholm Integral Equations of the Second Kind, *Applied Mathematics and Computation*, **148**:443-452:(2004).

[5] **Golbabai, A.** and **Keramati,B.** , Easy Computational approach to Solution of System of Linear Fredholm Integral Equations, *Chaos Solutions &Fractals* , **38**:568-574(2008).

[6] **Fox,L.** and **Parker,I.B**., *Chebyshev Polynomial in Numerical Analysis*, Oxford University Press,(1968).

[7] **De Bonie,M.C.** and **Laurita, C.,**Numerical Treatment of Second Kind Fredholm Integral Equations Systems on Bounded Intervals, *Journal of Computational and Applied Mathematics*, **217**: 64-87(2008).

[8] **Adomian,G.**,Solving Frontier Problems of Physics :The Decomposition method, *Kluwer, Academic Publishers* (1994).

دراسة مقارنة بين طريقة تشبشيف التجميعية وطريقة أدوميان للتجزئة لحل نظام خطي من معادلات فردهولم التكاملية من النوع الثاني

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المستخلص. في هذا البحث حصلنا على الحلول التقريبية لنظام من المعادلات التكاملية الخطية باستخدام طريقة كثيرات حدود تشيبشيف وكذلك باستخدام طريقة أدوميان للتجزئة. أيضا قمنا بتوضيح المقارنة بين الطريقتين.